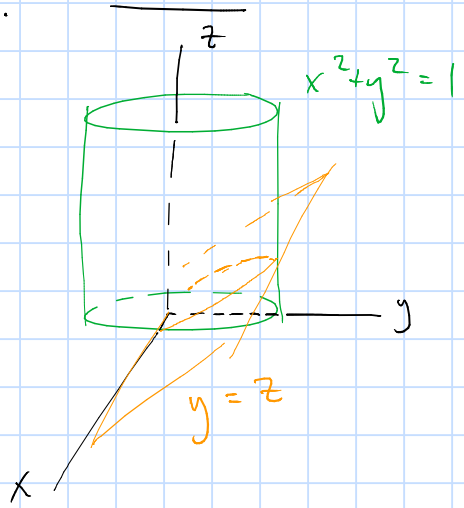
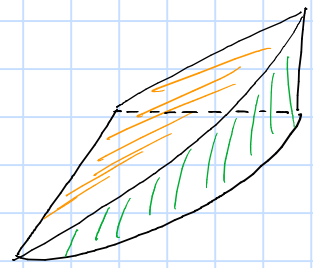
Ex: Compute the volume of the solid bounded by the cylinder $x^2+y^2=1$ and the plane $y=z$ in the first octant.

Sol:

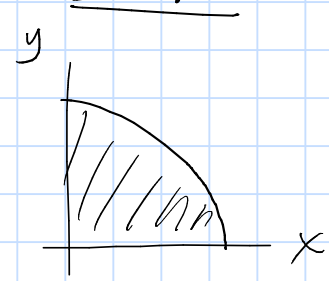
Picture



Cleaner Picture



Shadow



Now we parameterize the region R.

Cartesian Coords

$$R_{x,y,z} = \{(x,y,z) : 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2},\ 0 \le z \le y\}$$

Cylindrical Coords $\rightsquigarrow \begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$

$$R_{r,\theta,z} = \{(r,\theta,z) : 0 \le r \le 1,\ 0 \le \theta \le \tfrac{\pi}{2},\ 0 \le z \le r\sin(\theta)\}$$

Finally, we compute $\mathrm{Vol}(R) = \iiint_{R_{x,y,z}} 1 \, dV_{x,y,z}$ in two ways ← only one is necessary...

Cartesian Coords

$$\begin{aligned}
\mathrm{Vol}(R) &= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{y} 1 \, dz\, dy\, dx \\
&= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left[z\right]_{z=0}^{y} dy\, dx \\
&= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} y \, dy\, dx \\
&= \int_{x=0}^{1} \left[\tfrac{1}{2} y^2\right]_{y=0}^{\sqrt{1-x^2}} dx \\
&= \int_{x=0}^{1} \tfrac{1}{2}(1-x^2)\, dx \\
&= \tfrac{1}{2}\left[x - \tfrac{1}{3}x^3\right]_{x=0}^{1} \\
&= \tfrac{1}{2}\left(1 - \tfrac{1}{3} - 0\right) \\
&= \tfrac{1}{2} \cdot \tfrac{2}{3} = \tfrac{1}{3}
\end{aligned}$$

Cylindrical Coords

Jacobian!

$$\begin{aligned}
\mathrm{Vol}(R) &= \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{z=0}^{r\sin(\theta)} r \, dz\, d\theta\, dr \\
&= \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} r\left[z\right]_{z=0}^{r\sin(\theta)} d\theta\, dr \\
&= \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} r^2 \sin(\theta)\, d\theta\, dr \\
&= \int_{r=0}^{1} r^2 \left[-\cos(\theta)\right]_{\theta=0}^{\pi/2} dr \\
&= \int_{r=0}^{1} r^2 \left(-0+1\right) dr \\
&= \int_{r=0}^{1} r^2 \, dr \\
&= \left[\tfrac{1}{3} r^3\right]_{r=0}^{1} \\
&= \tfrac{1}{3}(1-0) = \tfrac{1}{3}
\end{aligned}$$